CLASSICAL MECHANICS: Worked examples

Michaelmas Term 2006

4 Lectures *Prof M. Brouard*

Lecture 1

EXAMPLE 1:

a) Velocity and acceleration as derivatives

Calculate the acceleration of a particle given its time dependent position:

$$\text{Position :} \qquad r_x(t) \;=\; \alpha t^3$$

where α is a constant and $r_0 = 0$ at $t = 0$.

$$\text{Velocity :} \qquad v_x(t) \;=\; \frac{\mathrm{d}r_x}{\mathrm{d}t} \;=\; 3\alpha t^2$$

where $v_0 = 0$.

$$\text{Acceleration :} \qquad a_x(t) \;=\; \frac{\mathrm{d}v_x}{\mathrm{d}t} \;=\; 6\alpha t$$

EXAMPLE 2:

b) Position and velocity as integrals

Calculate the position of a particle given its time dependent acceleration:

$$\text{Acceleration :} \qquad a_x(t) \left(\equiv \frac{\mathrm{d}v_x}{\mathrm{d}t}\right) \;=\; 6\alpha t$$

$v_x(t)$ is obtained by integration [1]

$$v_x(t) \;=\; \int a_x(t)\mathrm{d}t \;=\; \int 6\alpha t \;\mathrm{d}t \;=\; 3\alpha t^2 + \text{constant}$$

At $t = 0$ the velocity $v_0 = 0$, therefore

$$\text{Velocity :} \qquad v_x(t) \left(\equiv \frac{\mathrm{d}r_x}{\mathrm{d}t}\right) \;=\; 3\alpha t^2$$

Finally, the time dependent position is

$$\text{Position :} \quad r_x(t) \;=\; \int v_x(t)\,\mathrm{d}t = \int 3\alpha t^2 \mathrm{d}t \;=\; \alpha t^3$$

where we have assumed that $r_0 = 0$.

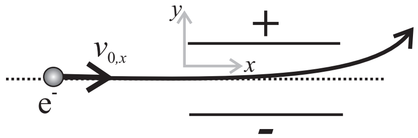
[1] This is an example of a first order *differential equation*

EXAMPLE 3:

Electrons, initially travelling at $2.4 \times 10^6\ m\,s^{-1}$ in the horizontal direction, enter a region between two horizontal charged plates of length 2 cm where they experience an acceleration of $4 \times 10^{14}\ m\,s^{-2}$, vertically upwards.

Find (a) the vertical position as they leave the region between the plates, and (b) the angle at which they emerge from between the plates.

For motion along the x coordinate



$$
\begin{aligned}
a_x &= 0 \\
v_x &= v_{0x} = 2.4 \times 10^6 \text{ m s}^{-1} \\
r_x &= r_{0x} + v_{0x}t \qquad\qquad r_{0x} = 0 \\
r_x &= 0.02 \text{ m} = 2.4 \times 10^6 t \\
t &= 8.33 \times 10^{-9} \text{ s}
\end{aligned}
$$

and along the y coordinate

$$
\begin{aligned}
a_y &= 4 \times 10^{14} \text{ m s}^{-2} \\
v_y &= v_{0y} + a_y t \qquad\qquad v_{0y} = 0 \\
r_y &= r_{0y} + \tfrac{1}{2} a_y t^2 \qquad\quad r_{0y} = 0
\end{aligned}
$$

Substituting for the time the electron spends between the plates

$$r_y = \tfrac{1}{2} a_y t^2 = 0.0139 \text{ m}$$

For the angle at which the electrons depart

$$\tan\theta = \frac{v_y}{v_x} = \frac{a_y t}{v_{0x}} \qquad\qquad \theta = 54.2^\circ$$

EXAMPLE 4:

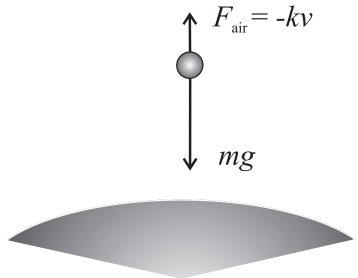
An object of mass m, dropped from height h, experiences a retarding force due to air resistance of kv, where k is a constant.

Assuming that gravity exerts a constant acceleration, g, what is the terminal velocity of the object, v_T?

Acceleration:

$$a \equiv \frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{k}{m}v$$

This *differential equation* can be solved by separating the variables, v and t, and integrating:



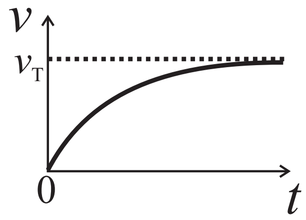
$$
\begin{aligned}
\int \frac{\mathrm{d}v}{g - \frac{k}{m}v} &= \int \mathrm{d}t + \text{constant} \\
-\frac{m}{k} \ln\left(g - \frac{k}{m}v\right) &= t + \text{constant} \\
\ln\left(g - \frac{k}{m}v\right) &= -\frac{k}{m}t + \text{constant}'
\end{aligned}
$$

When $t = 0$, $v = 0$

$$\ln g = \text{constant}'$$

Therefore

$$v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$



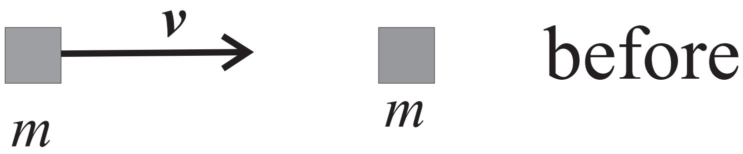
As $t \to \infty$, $v \to v_T$

$$v_T = \frac{mg}{k}$$

How does the height of the particle vary with time?

Lecture 2

EXAMPLE 5:



A particle of mass m travelling at a velocity v hits a stationary particle of the same mass and sticks to it. What is the final velocity, v_{f}, of the two particles after they collide and stick together?

Total initial momentum

$$p_{\mathrm{i}} = m_1 v_1 + m_2 v_2 = mv + 0 = mv$$

Total final momentum (after sticky collision)

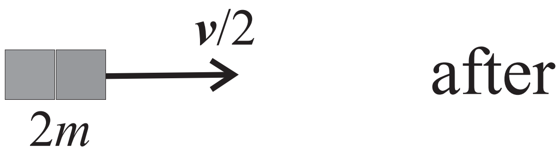
$$p_{\mathrm{f}} = (m_1 + m_2)\, v_{\mathrm{f}} = 2\, m v_{\mathrm{f}}$$

Conserving momentum

$$p_{\mathrm{i}} = p_{\mathrm{f}}$$

Hence

$$v_{\mathrm{f}} = \frac{m}{2m} v = \frac{1}{2} v$$



This is an example of an *inelastic collision*, i.e. one in which kinetic energy is not conserved.

EXAMPLE 6:

What constant force would be required to stop each of the following objects in 0.5 km: (a) a 150 g cricket ball moving at 40 m s^{-1}, (b) a 13 g bullet moving at 700 m s^{-1}, (c) a 1500 kg car moving at 200 km h^{-1}, and (d) a 1.8×10^5 kg airliner moving at 2240 km h^{-1}? Neglect the effects of gravity.

Use the work-energy theorem:

$$W = Fs = \Delta K$$

Because the final kinetic energy is zero,

$$\Delta K = K_{\mathrm{initial}} = \tfrac{1}{2} m v^2$$

i.e.

$$F = \frac{mv^2}{2s}$$

(a) 0.24 N

(b) 6.4 N

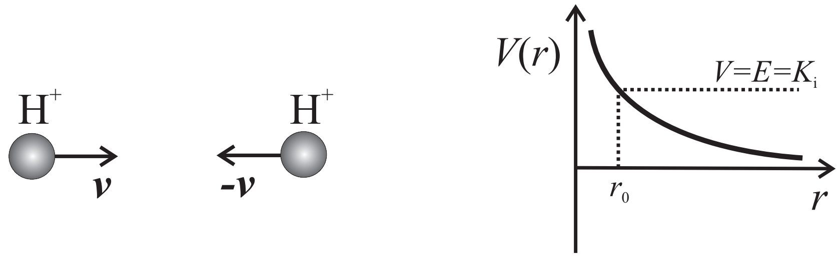
(c) 4.6×10^3 N

(d) 7.0×10^7 N

In this simple example, the force required scales with the kinetic energy of the particles.

EXAMPLE 7:

Two protons (H^+), initially separated at large r and possessing initial velocities $v = 200\,\mathrm{m\,s^{-1}}$, collide head-on. What is their separation of closest approach?



Assume that *initially* all the energy is kinetic energy, $E = K_i$, (since the protons are separated to large r initially). Because both protons have the same mass and speed we may write

$$K_i = E = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2 = mv^2$$

As the particles approach, the potential energy *increases* according to the equation

$$V(r) = \frac{q^2}{4\pi\epsilon_0 r}$$

and the kinetic energy *decreases* by the same amount (conservation of energy). At the separation of closest approach, r_0, both particles are momentarily stationary and thus

$$E = V(r_0) = mv^2$$

$$r_0 = \frac{q^2}{4\pi\epsilon_0 mv^2}$$

Lecture 3

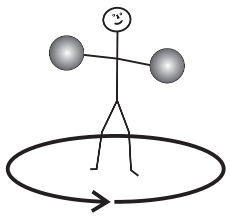
EXAMPLE 8:

Someone stands on a platform, which rotates at 0.5 revolutions s^{-1}. With arms outstretched, they hold two 4 kg blocks at a distance of 1 m from the axis of rotation, which passes through the centre of the person. They then reduce the distance of the blocks to 0.5 m.

Assuming that the moment of inertia of the person and the platform (excluding the blocks) is constant at 4 $kg\,m^2$, (a) what is the new angular velocity and (b) what is the change in kinetic energy?

(a) The moment of inertia of the system is initially

$$I_{\rm i} = 4 + 2mr_{\rm i}^2 = 12 \text{ kg m}^2$$



and finally

$$I_{\rm f} = 4 + 2mr_{\rm f}^2 = 6 \text{ kg m}^2$$

The initial angular frequency is $\omega_{\rm i} = 2\pi f_{\rm i} = \pi \text{ rad s}^{-1}$ and thus, applying angular momentum conservation,

$$I_{\rm i}\omega_{\rm i} = I_{\rm f}\omega_{\rm f}$$

$$\omega_{\rm f} = 2\pi \text{ rad s}^{-1}$$

(b) The initial and final kinetic energies are given by

$$K_{\rm i} = \tfrac{1}{2} I_{\rm i}\omega_{\rm i}^2 = 6\pi^2 \text{ J}$$

and

$$K_{\rm f} = \tfrac{1}{2} I_{\rm f}\omega_{\rm f}^2 = 12\pi^2 \text{ J}$$

Thus the change in kinetic energy is

$$\Delta K = K_{\rm f} - K_{\rm i} = 6\pi^2 \simeq 60 \text{ J}$$

Note that the increase in kinetic energy is supplied by the man doing work in pulling the blocks inwards.

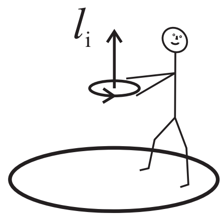
EXAMPLE 9:

A person stands on a stationary (but rotatable) platform with a spinning bicycle wheel in their hands. The moment of inertia of the person and the platform is $I_{\rm w} = 4$ $kg\,m^2$, and for the bicycle wheel it is $I_{\rm b} = 1$ $kg\,m^2$. The angular velocity is 10 rad s^{-1} counterclockwise as viewed from above.

Explain what happens when the person turns the wheel upside down (assuming there are no external torques acting on the system).

Define the z-axis pointing vertically upwards. The initial angular momentum is therefore

$$l_{\rm i} = +I_{\rm b}\,\omega_{\rm b}$$



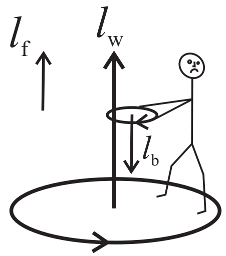
where the plus is defined by the right hand rule.

In the absence of external torques

$$l_{\rm f} = l_{\rm i}$$

Since the final angular momentum of the bicycle is $-I_{\rm b}\,\omega_{\rm b}$ the platform must rotate to conserve angular momentum.

$$l_{\rm f} = I_{\rm w}\omega_{\rm w} - I_{\rm b}\omega_{\rm b}$$



Equating the initial and final momenta one finds

$$\omega_{\rm w} = +2I_{\rm b}\omega_{\rm b}/I_{\rm w} = +5 \text{ rad s}^{-1}$$

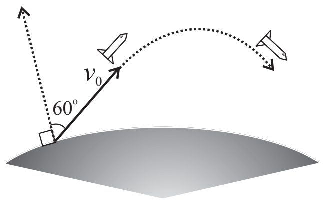
where the + sign indicates that the platform plus the person rotate in the counterclockwise direction (the + z direction).

EXAMPLE 10:

A rocket of mass m is fired at 60^o to the local vertical with an initial speed $v_0 = \sqrt{GM/R}$ where M and R are the mass and radius of the earth respectively. Show that its maximum distance from the earth's centre is $3R/2$.

The solution uses *both* conservation of energy and angular momentum. Initially, the angular momentum is

$$l \simeq mv_0 \sin 60\, R = m\sqrt{GMR} \sin 60$$



and the kinetic energy is

$$K = \tfrac{1}{2}mv_0^2 = GmM/2R$$

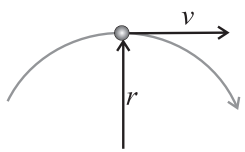
The total energy is therefore

$$E = K+V = K-GmM/R = -GmM/2R$$

At the maximum height, r, the rocket is no longer moving radially outwards. However, to conserve angular momentum, it must still have angular kinetic energy given by

$$K_{\text{ang}} = l^2/2I = 3mGMR/8r^2$$

Thus the total energy at the maximum height is



$$E = V + K_{\text{ang}} = -GmM/r + 3GmMR/8r^2$$

Equating the two expressions for the total energy yields

$$-\frac{1}{2R} = -\frac{1}{r} + \frac{3R}{8r^2}$$

$$4r^2 - 8Rr + 3R^2 = 0$$

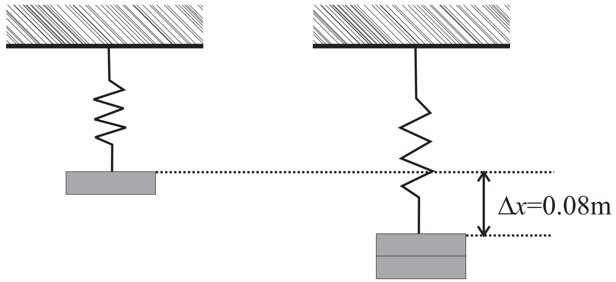
$$r = \frac{8R \pm 4R}{8} = \frac{3}{2}R \text{ or } \frac{1}{2}R$$

The latter solution is unphysical!

Lecture 4

EXAMPLE 11:

A mass of 100 g hangs at the end of a spring (which obeys Hooke's Law). When the mass is increased by another 100 g the spring extends a further 8 cm. What is the spring constant? (Assume that the acceleration due to gravity is 9.8 m s⁻².)



Because the mass is at equilibrium in both cases the net force on the mass is zero, and we can write

$$F = 0 \qquad \text{i.e.} \quad kx = mg$$

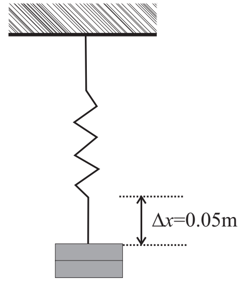
With extra mass added

$$F = 0 \qquad kx' = k(x + 0.08) = 2mg$$

Subtracting the two expressions obtained before and after addition of the extra mass gives

$$k = \frac{mg}{\Delta x} = 12.25\,\mathrm{Nm}^{-1}$$

If the masses are pulled down a further 5 cm and then released, what would be (i) the frequency of vibration, (ii) the maximum velocity of the masses, and (iii) the maximum acceleration of the resulting motion.



(i) From the definition of the frequency of a harmonic oscillator

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{12.25}{0.2}} = 7.8\,\mathrm{rad\,s}^{-1}$$

$$f = \frac{1}{2\pi}\omega = 1.25\,\mathrm{Hz}$$

(ii) The velocity of the harmonic oscillator (the time derivative of the position) is

$$v(t) = A\omega\cos(\omega t + \phi)$$

with $A = 0.05\,\mathrm{m}$. This is a maximum (minimum) when $\cos(z) = \pm 1$, i.e. when $z = 0, \pi$,

$$v_{\mathrm{max}} = \pm 0.05\,\omega = \pm 0.39\,\mathrm{m\,s}^{-1}$$

Note that the position (displacement from equilibrium) is zero at this time.

(iii) Similarly, the maximum acceleration is

$$a_{\mathrm{max}} = -A\omega^2 = \mp 3.0\,\mathrm{m\,s}^{-2}$$

which is obtained by taking the maximum in the time derivative of the velocity.

EXAMPLE 12:

Show that the function

$$x = \alpha \mathrm{e}^{i\omega t} + \beta \mathrm{e}^{-i\omega t} \tag{1}$$

is a solution of the differential equation describing a harmonic oscillator of frequency ω:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x$$

Comment on the relationship between the above equation for x and the solution $x = A\sin(\omega t + \phi)$.

Taking the first and second time derivative of x one obtains

$$\dot{x} = i\omega(\alpha \mathrm{e}^{i\omega t} - \beta \mathrm{e}^{-i\omega t})$$

$$\ddot{x} = i^2\omega^2(\alpha \mathrm{e}^{i\omega t} + \beta \mathrm{e}^{-i\omega t})$$

The latter can be written

$$\ddot{x} = -\omega^2 x$$

and therefore equation (1) clearly satisfies the differential equation (it is actually the general solution).

The link with the more familiar solution $x = A\sin(\omega t + \phi)$ can be established using the fact that

$$\mathrm{e}^{\pm i\omega t} = \cos\omega t \pm i \, \sin\omega t$$

Substitution of this equation in the top expression for x yields

$$x = C\cos\omega t + D\sin\omega t \tag{2}$$

with $C = \alpha + \beta$ and $D = i(\alpha - \beta)$. However, using standard trigonometry $x = A\sin(\omega t + \phi)$ can be rewritten

$$x = A\left(\sin\omega t\cos\phi + \cos\omega t\sin\phi\right)$$

which is the same as equation (2) with $C = A\cos\phi$ and $D = A\sin\phi$.